

# Quantum Criticality and Superconductivity in Quasi-Two-Dimensional Dirac Electronic Systems

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## Abstract

We present a theory describing the superconducting (SC) interaction of Dirac electrons in a quasi-two-dimensional system consisting of a stack of  $N$  planes. The occurrence of a SC phase is investigated both at  $T = 0$  and  $T \neq 0$ , in the case of a local interaction, when the theory must be renormalized and also in the situation where a natural physical cutoff is present in the system. In both cases, at  $T = 0$ , we find a quantum phase transition connecting the normal and SC phases at a certain critical coupling. The phase structure is shown to be robust against quantum fluctuations. The SC gap is determined for  $T = 0$  and  $T \neq 0$ , both with and without a physical cutoff and the interplay between the gap and the SC order parameter is discussed. Our theory qualitatively reproduces the SC phase transition occurring in the underdoped regime of the high- $T_c$  cuprates. This fact points to the possible relevance of Dirac electrons in the mechanism of high- $T_c$  superconductivity.

*Key words:* Dirac electrons, superconductivity, quantum criticality

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## 1 Introduction

Surprisingly, there are many condensed matter systems in one and two spatial dimensions containing electrons that may be described by a relativistic, Dirac-type lagrangian, namely Dirac electrons. Even though these are evidently non-relativistic systems, this fact occurs because in some materials there are special points in the Brillouin zone where two bands touch in a single point around which the electron dispersion relation may be linearized as  $\epsilon(\vec{k}) = v_F |\vec{k}|$ . The kinematics of these electrons can be described by a Dirac-type lagrangian where the velocity  $v_F$  determines the angle of the Dirac cone. At the tip of this cone the Fermi surface reduces to a point, the Fermi point, and the density

of states vanishes. The elementary excitations around a Dirac point are Dirac electrons. They are, after all, a result of the electron-lattice interaction.

There are many important quasi-two-dimensional systems containing Dirac electrons. Among them we could mention the high-Tc cuprates, where Dirac points appear in the intersection of the nodes of the d-wave superconducting gap and the 2D-Fermi surface. Because of these nodes the low-energy quasiparticle spectrum is gapless and the dispersion relation can be linearized as described above [1]. The low-energy quasiparticle dynamics is determined exclusively by these points, since they are occupied even at very low temperatures [2,3,4,5]. For these reasons, Dirac electrons are expected to play an important role in the cuprates. This motivates the application of the model introduced below to these materials and the results thereby obtained seem to point towards this direction.

Dirac electrons also appear in semi-metals such as graphene sheets or stacks thereof, namely graphite, where the vanishing density of states at the Fermi points has important consequences in the electronic properties like, for instance, the absence of screening of the Coulomb potential [6,7,8,9,10,11].

Another class of materials where the presence of Dirac electrons produces interesting effects are the rare-earth dichalcogenides such as  $2H-TaSe_2$ ,  $2H-NbSe_2$ ,  $2H-TaS_2$  and  $2H-NbS_2$ . In these systems charge-density-wave order coexists with superconductivity at low temperatures and Dirac points form in the intersection of the Fermi surface with the nodal lines of the charge-density-wave order parameter [12,13]. This is a particular example of nodal liquids, which in general contain Dirac electrons [14]. Finally, we may list carbon nanotubes as another important type of materials that have also been shown to possess Dirac electrons [15].

In this work, we present a theory describing the superconducting interaction of Dirac electrons associated to two distinct Dirac points belonging to a stack of  $N$  planes. We analyze the conditions for the existence of a superconducting gap, both at  $T = 0$  and  $T \neq 0$ , either for finite  $N$  or in the limit  $N \rightarrow \infty$ , which corresponds to the case when the system is actually three-dimensional.

At  $T = 0$ , we show that the system presents a quantum critical point separating the normal and superconducting phases and determine the superconducting gap as a function of the coupling constant. The theory is renormalized in a  $1/N$  expansion. A renormalization group analysis is then performed, demonstrating the independence of physical quantities from the renormalization point. We also investigate the effect of quantum fluctuations in our results and demonstrate that the phase diagram found in mean field is robust against these fluctuations.

We then consider the finite temperature case and determine the superconduct-

ing gap  $\Delta$  as a function of the temperature and of the zero temperature gap. We find a critical temperature  $T_c$ , where the gap vanishes. The possibility of occurrence of dynamical generation of a superconducting gap without the corresponding spontaneous breakdown of the U(1) symmetry, in compliance with the Coleman-Mermin-Wagner-Hohenberg [16] theorem is discussed in detail, as well as the associated Kosterlitz-Thouless transition suffered by the phase of the complex order parameter.

We finally make a detailed study of the situation usually found in condensed matter applications, where a natural momentum cutoff exists in the system, both at  $T = 0$  and  $T \neq 0$ . We consider the general regime, where the cutoff is not necessarily much larger than the gap, as well as the weak coupling regime where the cutoff is much larger than the gap. The former situation is likely to be relevant for high- $T_c$  superconductors while the latter is the one found in conventional BCS superconductors.

The quantum phase transition occurring in our model and the behavior of  $T_c$  around the quantum critical point qualitatively reproduce very well the superconducting transition in the high- $T_c$  cuprates in the underdoped region. This suggests that Dirac electrons may play an important role in the mechanism of high- $T_c$  superconductivity.

## 2 The Model and the Effective Action

We consider a quasi-two-dimensional electronic system consisting of a stack of planes containing two Dirac points. In addition, we introduce an internal index  $a = 1, \dots, N$ , supposed to characterize the different planes to which the electrons may belong. The electron creation operator, therefore, is given by  $\psi_{i\sigma a}^\dagger$ , where  $i = 1, 2$  are the Dirac indices, corresponding to the two Fermi points,  $\sigma = \uparrow, \downarrow$ , specifies the z-component of the electron spin and  $a = 1, \dots, N$  labels the electron plane. We assume, further, that there is a BCS-type superconducting interaction, whose origin is understood to be determined by some underlying microscopic theory. In the case of the high- $T_c$  cuprates, in particular, there is no consensus about what would be such a theory, however it is generally accepted that superconductivity is constrained to the  $CuO_2$  planes.

The complete lagrangian we will consider is given by

$$\mathcal{L} = i\bar{\psi}_{\sigma a} \not{\partial} \psi_{\sigma a} + g \left( \psi_{1\uparrow a}^\dagger \psi_{2\downarrow a}^\dagger + \psi_{2\uparrow a}^\dagger \psi_{1\downarrow a}^\dagger \right) (\psi_{2\downarrow b} \psi_{1\uparrow b} + \psi_{1\downarrow b} \psi_{2\uparrow b}), \quad (1)$$

where  $g > 0$  is a constant that may depend on some external control parameter, such as the pressure or the concentration of some dopant. Later on, we will make  $g \equiv \frac{\lambda}{N}$ .

We use the following convention for the Dirac matrices:

$$\gamma^0 = \sigma^z, \quad \gamma^0 \gamma^1 = \sigma^x, \quad \gamma^0 \gamma^2 = \sigma^y, \quad (2)$$

Observe that the interaction lagrangian contains four terms, describing the various possible BCS interactions in which a Cooper pair would form between electrons with opposite spins, belonging to different Fermi points but in the same plane. Nevertheless, the interaction of electrons belonging to different planes is allowed.

Furthermore, we can motivate our model out of a microscopic description of the high-Tc cuprates by noting that t-J model calculations of the electron spectral function indicate the emergence of small pockets of Dirac electrons at low doping [17].

In addition to a complex valued O(N) symmetry [18], the lagrangian above possesses the U(1) symmetry

$$\psi_{i\sigma a} \rightarrow e^{i\theta} \psi_{i\sigma a} \quad \psi_{i\sigma a}^\dagger \rightarrow e^{-i\theta} \psi_{i\sigma a}^\dagger \quad i = 1, 2 \quad (3)$$

and the chiral U(1) symmetry

$$\psi_{1\sigma a} \rightarrow e^{i\theta} \psi_{1\sigma a} \quad \psi_{2\sigma a} \rightarrow e^{-i\theta} \psi_{2\sigma a}. \quad (4)$$

We shall see below that the former symmetry is spontaneously broken at  $T = 0$ . A model presenting the spontaneous breakdown of the latter has been studied in [19].

We now introduce a Hubbard-Stratonovitch complex scalar field  $\sigma$  through

$$\begin{aligned} \mathcal{Z}_\sigma &= \int \mathcal{D}\sigma^* \mathcal{D}\sigma \exp \left\{ -i \int d^3x \frac{1}{g} \sigma^* \sigma \right\} \\ &= \int \mathcal{D}\sigma^* \mathcal{D}\sigma \exp \left\{ -i \int d^3x \frac{1}{g} \left[ \sigma^* - g \left( \psi_{1\uparrow a}^\dagger \psi_{2\downarrow a}^\dagger + \psi_{2\uparrow a}^\dagger \psi_{1\downarrow a}^\dagger \right) \right] \right. \\ &\quad \left. \times \left[ \sigma - g \left( \psi_{2\downarrow b} \psi_{1\uparrow b} + \psi_{1\downarrow b} \psi_{2\uparrow b} \right) \right] \right\}. \end{aligned} \quad (5)$$

In terms of this, we may write the partition function

$$\mathcal{Z} = \frac{1}{\mathcal{Z}_\Psi^0} \int \mathcal{D}\Psi^\dagger \mathcal{D}\Psi \mathcal{D}\sigma \exp \left\{ i \int d^3x \mathcal{L} \right\} \quad (6)$$

as

$$\mathcal{Z} = \frac{1}{\mathcal{Z}_\sigma \mathcal{Z}_\Psi^0} \int \mathcal{D}\Psi^\dagger \mathcal{D}\Psi \mathcal{D}\sigma^* \mathcal{D}\sigma \exp \left\{ i \int d^3x \mathcal{L}[\Psi, \sigma] \right\}, \quad (7)$$

where

$$\begin{aligned}\mathcal{L}[\Psi, \sigma] = & i\bar{\psi}_{\sigma a} \not{\partial} \psi_{\sigma a} - \frac{1}{g} \sigma^* \sigma - \sigma^* (\psi_{2\downarrow b} \psi_{1\uparrow b} + \psi_{1\downarrow b} \psi_{2\uparrow b}) \\ & - \sigma (\psi_{1\uparrow a}^\dagger \psi_{2\downarrow a}^\dagger + \psi_{2\uparrow a}^\dagger \psi_{1\downarrow a}^\dagger).\end{aligned}\quad (8)$$

From this we obtain the field equation for the auxiliary field  $\sigma$ :

$$\sigma = -g (\psi_{2\downarrow a} \psi_{1\uparrow a} + \psi_{1\downarrow a} \psi_{2\uparrow a}) \quad (9)$$

As we shall see, the vacuum expectation value of  $\sigma$  will be taken as the order parameter for the superconducting phase.

We will now integrate over the fermionic fields. In order to do that, we introduce the Nambu fermion field  $\Phi_a^\dagger = (\psi_{1\uparrow a}^\dagger \psi_{2\uparrow a}^\dagger \psi_{1\downarrow a}^\dagger \psi_{2\downarrow a}^\dagger)$ . In terms of this we can rewrite (8) as

$$\mathcal{L}[\Psi, \sigma] = -\frac{1}{g} \sigma^* \sigma + \Phi_a^\dagger \mathcal{A} \Phi_a, \quad (10)$$

where the matrix  $\mathcal{A}$  is given, in momentum space, by

$$\mathcal{A} = \begin{pmatrix} -k_0 & k_- & 0 & -\sigma \\ k_+ & -k_0 & -\sigma & 0 \\ 0 & -\sigma^* & -k_0 & -k_+ \\ -\sigma^* & 0 & -k_- & -k_0 \end{pmatrix} \quad (11)$$

with  $k_\pm = v_F (k_2 \pm i k_1)$ .

Integrating on the fermion fields and redefining the coupling constant as  $g = \frac{\lambda}{N}$ , we obtain

$$\mathcal{Z} = \frac{1}{\mathcal{Z}_\sigma} \int \mathcal{D}\sigma^* \mathcal{D}\sigma e^{iS_{eff}[\sigma]}, \quad (12)$$

where

$$S_{eff}[\sigma] = \int d^3x \left( -\frac{N}{\lambda} |\sigma|^2 \right) - iN \ln \text{Det} \left[ \frac{\mathcal{A}[\sigma]}{\mathcal{A}[0]} \right]. \quad (13)$$

The determinant of the matrix  $\mathcal{A}$  is  $\det \mathcal{A}[\sigma] = [(k_0^2 - v_F^2 |\vec{k}|^2) - |\sigma|^2]^2$ , hence the above expression becomes

$$S_{eff}[\sigma] = \int d^3x \left( -\frac{N}{\lambda} |\sigma|^2 \right) - i2N \text{Tr} \ln \left[ 1 + \frac{|\sigma|^2}{\square} \right] \quad (14)$$

### 3 The Superconducting Transition at $T = 0$ and Quantum Criticality

Let us consider in this section the  $T = 0$  case. We shall see that a quantum phase transition occurs, connecting the superconducting and normal phases.

#### 3.1 The Renormalized Effective Potential

Since we are considering the zero temperature case, the functional integral in (12) must be dominated by constant configurations of  $\sigma$ , which minimize the effective potential per plane,  $V_{\text{eff}}$ , corresponding to (14). This is more conveniently evaluated in the euclidean space and is given by

$$V_{\text{eff}}(|\sigma|) = \frac{|\sigma|^2}{\lambda} - 2 \int \frac{d^2 k}{(2\pi)^2} \int \frac{d\omega}{2\pi} \left\{ \ln \left[ 1 + \frac{|\sigma|^2}{\omega^2 + v_F^2 k^2} \right] \right\}, \quad (15)$$

where, henceforth, by  $\sigma$  we actually mean  $\langle 0|\sigma|0\rangle$ . The above expression corresponds to a mean field approximation. Conversely, this would be the leading order approximation in an  $1/N$  expansion and would be the exact result for  $N \rightarrow \infty$ .

The explicit form of the effective potential may be evaluated in this framework, by introducing a large momentum cutoff  $\Lambda/v_F$ . The resulting expression, obtained from (15) is

$$V_{\text{eff}}(|\sigma|) = \frac{|\sigma|^2}{\lambda} - \frac{\Lambda}{2\pi v_F^2} |\sigma|^2 + \frac{|\sigma|^3}{3\pi v_F^2}. \quad (16)$$

Quartic fermionic theories in 2+1D have been shown to be renormalizable in a  $1/N$  expansion [20]. In order to eliminate the divergent constant  $\Lambda$  from (16) we renormalize the coupling constant  $\lambda$  using the usual renormalization condition [21]

$$\left. \frac{\partial^2 V_{\text{eff}}}{\partial \sigma \partial \sigma^*} \right|_{|\sigma|=\sigma_0} = \frac{1}{\lambda_R}, \quad (17)$$

where  $\sigma_0$  is an arbitrary finite scale parameter, the renormalization point and  $\lambda_R$  is the (finite), renormalized coupling constant.

Inserting (16) in (17), we obtain

$$\frac{1}{\lambda_R} = \frac{1}{\lambda} - \frac{\Lambda}{\alpha} + \frac{3\sigma_0}{2\alpha}, \quad (18)$$

where  $\alpha \equiv 2\pi v_F^2$ .

Substituting this result in (16), we get the renormalized effective potential per plane

$$V_{\text{eff},R}(|\sigma|) = \frac{|\sigma|^2}{\lambda_R} - \frac{3\sigma_0}{2\alpha}|\sigma|^2 + \frac{2}{3\alpha}|\sigma|^3. \quad (19)$$

### 3.2 The Gap Equation and the Quantum Critical Point

Let us study now the minima of the renormalized effective potential per plane, Eq. (19). The first and second derivatives of  $V_{\text{eff},R}$  with respect to  $|\sigma|$  are given, respectively, by

$$V'_{\text{eff},R}(|\sigma|) = 2|\sigma| \left( \frac{1}{\lambda_R} - \frac{3\sigma_0}{2\alpha} + \frac{|\sigma|}{\alpha} \right) \quad (20)$$

and

$$V''_{\text{eff},R}(|\sigma|) = 2 \left( \frac{1}{\lambda_R} - \frac{3\sigma_0}{2\alpha} + \frac{2|\sigma|}{\alpha} \right) \quad (21)$$

The ground state is determined by the solutions of  $V'_{\text{eff},R} = 0$ . This admits two solutions, which we call  $\Delta$ . Notice that

$$\Delta = |\langle 0|\sigma|0\rangle| \quad (22)$$

and  $\langle 0|\sigma|0\rangle$  is a complex order parameter for superconductivity.

From (20) we conclude that either  $\Delta = 0$  or  $\Delta \neq 0$ , the nonzero solutions satisfying the gap equation

$$1 = \frac{\lambda_R}{\alpha} \left( \frac{3\sigma_0}{2} - \Delta \right). \quad (23)$$

Inserting the  $\Delta = 0$  solution in (21), we get

$$V''_{\text{eff},R}(\Delta = 0) = 2 \left( \frac{1}{\lambda_R} - \frac{3\sigma_0}{2\alpha} \right) \quad (24)$$

and we conclude that  $\Delta = 0$  will be a minimum only for  $\lambda_R < 2\alpha/3\sigma_0 \equiv \lambda_c$ .

From (23), conversely, we see that the  $\Delta \neq 0$  solution is given by

$$\Delta_0 = \alpha \left( \frac{1}{\lambda_c} - \frac{1}{\lambda_R} \right). \quad (25)$$

Since  $\Delta_0$  is positive semi-definite, the above expression will actually be a solution only for  $\lambda_R > \lambda_c$ . On the other hand, from (21), we immediately see that  $V''_{\text{eff},R}(\Delta) = 2\Delta_0/\alpha > 0$  for the solutions of the gap equation (23).

We can infer that the ground state of the system will be

$$\Delta_0 = \begin{cases} 0 & \lambda_R < \lambda_c \\ \alpha \left( \frac{1}{\lambda_c} - \frac{1}{\lambda_R} \right) & \lambda_R > \lambda_c \end{cases}. \quad (26)$$

Expression (26) implies that the system undergoes a continuous quantum phase transition at the quantum critical point  $\lambda_c = 4\pi v_F^2/3\sigma_0$ . Since  $\Delta$  is the modulus of the order parameter for superconductivity, we conclude that, for couplings below  $\lambda_c$  the electronic system will be in the normal state, while for couplings above  $\lambda_c$ , it will be in a superconducting one (for temperature effects, see the next section). The quantum critical point  $\lambda_c$ , therefore, separates a normal from a superconducting phase.

### 3.3 Renormalization Group Analysis

Let us now apply renormalization group methods, in order to show that our results are completely independent of the arbitrary finite scale  $\sigma_0$  introduced in our renormalization procedure.

We start showing that the renormalized effective potential  $V_{\text{eff},R}$ , given by (19) satisfies a renormalization group equation. Indeed it is easy to see that

$$\left( \sigma_0 \frac{\partial}{\partial \sigma_0} + \beta \frac{\partial}{\partial \lambda_R} \right) V_{\text{eff},R} = 0 \quad (27)$$

with the  $\beta$ -function given by  $\beta = -3\sigma_0/2\alpha \lambda_R^2 = -\lambda_R^2/\lambda_c$ . This means that the renormalized effective potential does not depend on the renormalization point  $\sigma_0$ . A negative  $\beta$ -function, on the other hand, implies that the theory is asymptotically free. Indeed, keeping  $\Delta_0$  fixed and taking the limit  $\sigma_0 \rightarrow \infty$  we see that  $\lambda_R \rightarrow 0$ .

One can also show that the gap  $\Delta_0$ , given by (25) satisfies the same renormalization group equation being, therefore, also independent of  $\sigma_0$ . Indeed, using (25) and the expression of the  $\beta$ -function just found, we obtain

$$\left( \sigma_0 \frac{\partial}{\partial \sigma_0} + \beta \frac{\partial}{\partial \lambda_R} \right) \Delta_0 = 0. \quad (28)$$

Finally, solving the differential equation corresponding to the  $\beta$ -function def-



inition, namely,

$$\sigma_0 \frac{\partial \lambda_R}{\partial \sigma_0} = \beta, \quad (29)$$

we obtain

$$\frac{1}{\lambda_R(\sigma'_0)} - \frac{1}{\lambda_c(\sigma'_0)} = \frac{1}{\lambda_R(\sigma''_0)} - \frac{1}{\lambda_c(\sigma''_0)}, \quad (30)$$

where  $\frac{1}{\lambda_c(\sigma'_0)} = \frac{3\sigma'_0}{2\alpha}$  and  $\sigma'_0$  and  $\sigma''_0$  are two arbitrary values of the renormalization point. The above equation, clearly shows that the superconducting gap  $\Delta_0$ , given by (25) is independent of the scale  $\sigma_0$ .

The theory predicts the existence of a quantum critical point  $\lambda_c$ , separating two phases at  $T = 0$ : a normal ( $\Delta_0 = 0$ ) and a superconducting one ( $\Delta_0 \neq 0$ ). Nevertheless, the theory does not predict the value of  $\lambda_c$ . This has to be determined experimentally. The renormalization group analysis, however, guarantees that the physics of the quantum phase transition will not depend on the renormalization point  $\sigma_0$ .

### 3.4 Gaussian Quantum Fluctuations

Let us now investigate whether the results we found in the saddle-point approximation for the ground state of the fermionic system described by (1) are robust against quantum fluctuations at  $T = 0$ . Notice that for the actual existence of the quantum phase transition it is necessary that the normal phase found in the saddle point approximation should not be removed by higher order corrections, as it happens, for instance, in the Coleman-Weinberg mechanism [21]. For this it is crucial that the corrected gap equation admits a  $\Delta_0 = 0$  solution.

Expanding the effective action (14) in (12) about a stationary point  $\sigma$  and retaining the gaussian quantum fluctuations  $\eta$ , we obtain, after integrating over  $\eta$  and  $\eta^*$ ,

$$\tilde{S}_{\text{eff}}[\sigma] = S_{\text{eff}}[\sigma] - \text{Tr} \ln \mathcal{M}[\sigma], \quad (31)$$

where  $S_{\text{eff}}[\sigma]$  is given by (14) and

$$\mathcal{M} = \begin{pmatrix} \frac{\delta^2 S}{\delta \sigma \delta \sigma^*} & \frac{\delta^2 S}{\delta \sigma^2} \\ \frac{\delta^2 S}{\delta \sigma^{*2}} & \frac{\delta^2 S}{\delta \sigma^* \delta \sigma} \end{pmatrix} [\sigma] \quad (32)$$

The effective potential corresponding to this is given by

$$\tilde{V}_{\text{eff}}(|\sigma|) = V_{\text{eff}}(|\sigma|) + \mathcal{V}(|\sigma|), \quad (33)$$

where  $V_{\text{eff}}(|\sigma|)$  is given by (16) and

$$\mathcal{V}(|\sigma|) = - \int \frac{d^2 k}{(2\pi)^2} \int \frac{d\omega}{2\pi} \left\{ \ln \left[ \frac{1}{\lambda} - a(\omega, \vec{k}) \right] + \ln \left[ \frac{1}{\lambda} - a(\omega, \vec{k}) + 2|\sigma|^2 b(\omega, \vec{k}) \right] \right\}, \quad (34)$$

with

$$a(\omega, \vec{k}, |\sigma|) = \int \frac{d^2 q}{(2\pi)^2} \int \frac{d\theta}{2\pi} \frac{2}{[v_F |\vec{q}| + i\theta] [v_F |\vec{q} + \vec{k}| - i(\theta + \omega)] + |\sigma|^2} \quad (35)$$

and  $b(\omega, \vec{k}, |\sigma|) = -\frac{\partial a}{\partial |\sigma|^2}$ .

We would like to stress at this point that, for the expansion about the saddle-point to make sense, we must have  $\mathcal{V}(\sigma) \ll V_{\text{eff}}(\sigma)$ . This condition is actually usually satisfied because the expansion is a power series in  $\hbar$ . Keeping this fact in mind, let us look for the minima of  $\tilde{V}_{\text{eff}}(\sigma)$ . From (34), we can infer that the first derivative of  $\mathcal{V}(\sigma)$  is of the form

$$\mathcal{V}'(|\sigma|) = |\sigma| f(|\sigma|), \quad (36)$$

where  $f(0)$  is a finite constant. Combining (36) with (20), we immediately conclude that  $\Delta_0 = 0$  is a solution of the corrected gap equation  $\tilde{V}'_{\text{eff}}(\sigma) = 0$ . The second derivative of the corrected effective potential  $\tilde{V}_{\text{eff}}(\sigma)$ , evaluated at this point is

$$\tilde{V}_{\text{eff}}''(\Delta = 0) = 2 \left( \frac{1}{\lambda_R} - \frac{3\sigma_0}{2\alpha} \right) + \mathcal{V}''(\Delta_0 = 0), \quad (37)$$

where we have used (24). Since  $\frac{\mathcal{V}''}{V_{\text{eff}}''} \propto \hbar^2$ , it follows that the second term in the rhs of the above equation must be much smaller than the first one. Therefore the sign of  $\tilde{V}_{\text{eff}}''(\sigma = 0)$  must be the same as the one of  $V_{\text{eff}}''(\sigma = 0)$ . We conclude that  $\Delta_0 = 0$  is indeed a minimum, even considering the quantum fluctuations. For the same reason, the mean field solution  $\Delta_0 \neq 0$  should not be cancelled by quantum corrections. Thus, we conclude that the phase structure we found at  $T = 0$  is robust against quantum fluctuations.

## 4 The Superconducting Transition at $T \neq 0$

### 4.1 The Gap Equation at $T \neq 0$

We consider in this section the finite temperature effects in the superconducting transition and in the order parameter  $\Delta$ . The nonzero solutions for  $\Delta$  at a finite temperature are supposed to hold *a priori* only in the  $N \rightarrow \infty$

limit, because otherwise they are ruled out by the Coleman-Mermin-Wagner-Hohenberg theorem [16]. This limit corresponds to a physical situation where the three-dimensionality of the system is explicitly taken in account. For finite values of  $N$  and  $T \neq 0$ , the situation is quite subtle. We discuss it in the next subsection.

At  $T \neq 0$ , the effective potential is no longer given by (15). It must be replaced by

$$V_{\text{eff}}(|\sigma|, T) = \frac{|\sigma|^2}{\lambda} - 2T \int \frac{d^2k}{(2\pi)^2} \sum_{n=-\infty}^{\infty} \left\{ \ln \left[ 1 + \frac{|\sigma|^2}{\omega_n^2 + v_F^2 k^2} \right] \right\}, \quad (38)$$

where  $\omega_n = (2n+1)\pi T$  are the fermionic Matsubara frequencies corresponding to the functional integration over the electron fields.

The finite temperature corrections do not alter the ultraviolet divergence structure of the theory, hence we may eliminate the divergences in the  $T \neq 0$  case through the same renormalization of the coupling constant  $\lambda$  as in the zero temperature case, given by (18).

In order to derive the gap equation, we consider the following condition, which must be satisfied by the order parameter at a finite temperature:

$$V'_{\text{eff}}(|\sigma|, T) = 2|\sigma| \left\{ \frac{1}{\lambda} - \frac{1}{2\alpha} \int_0^{\Lambda^2} dx \frac{1}{\sqrt{x + |\sigma|^2}} \tanh \left( \frac{\sqrt{x + |\sigma|^2}}{2T} \right) \right\} = 0. \quad (39)$$

This was obtained by taking the derivative of (38) with respect to  $|\sigma|$  and performing the Matsubara sum. In the above equation,  $\Lambda$  is the same cutoff used before.

As in the  $T = 0$  case, this admits two solutions, either  $\Delta(T) = 0$  or  $\Delta(T) \neq 0$ . In the latter case, the superconducting order parameter satisfies the gap equation

$$1 = \frac{\lambda}{\alpha} \int_{\Delta}^{\Lambda} dy \tanh \left( \frac{y}{2T} \right). \quad (40)$$

Solving the integral and renormalizing the coupling  $\lambda$  as in (18), we find a general expression for the superconducting gap as a function of the temperature, namely

$$\Delta(T) = 2T \cosh^{-1} \left[ \frac{e^{\frac{\Delta_0}{2T}}}{2} \right], \quad (41)$$

where  $\Delta_0$  is given by (26), see Fig. 1. From (41) we can verify that indeed  $\Delta(T = 0) = \Delta_0$ . Also from the above equation, we may determine the critical temperature  $T_c$  for which the superconducting gap vanishes. Using the fact

that  $\Delta(T_c) = 0$ , we readily find from (41)

$$T_c = \frac{\Delta_0}{2 \ln 2}. \quad (42)$$

This relation has been previously found in systems with a natural cutoff in the weak coupling regime [12] (see next section). It has also appeared in systems with dynamical mass generation in 2+1D [22,23,24,25].

In Fig. 2, using (26) and (42), we display  $T_c$  as a function of the coupling constant. This qualitatively reproduces the superconducting phase transition of the high- $T_c$  cuprates in the underdoped region. Since our theory describes the generic superconducting interaction of two-dimensional Dirac electrons, we may see this result as an indication of the possible relevance of this type of electrons in the high- $T_c$  mechanism.

In terms of the critical temperature, we may also express the gap as

$$\Delta(T) = 2T \cosh^{-1} \left[ 2 \left( \frac{T_c}{T} - 1 \right) \right]. \quad (43)$$

Near  $T_c$ , this yields

$$\Delta(T) \stackrel{T \lesssim T_c}{\approx} 2\sqrt{2 \ln 2} T_c \left( 1 - \frac{T}{T_c} \right)^{\frac{1}{2}}, \quad (44)$$

which presents the typical mean field critical exponent 1/2.

We would like to remark, finally, that both the gap  $\Delta(T)$  and the critical temperature do not depend on the arbitrary renormalization point  $\sigma_0$ .

## 4.2 Dynamical Gap Generation versus Spontaneous Symmetry Breaking

The well-known Coleman-Mermin-Wagner-Hohenberg theorem [16] forbids the occurrence of spontaneous breakdown of a continuous symmetry at a nonzero temperature for systems in two spatial dimensions. Our superconducting order parameter is complex and given by

$$\langle 0 | \sigma | 0 \rangle = \Delta e^{i\theta}, \quad (45)$$

where  $\Delta$  is the gap. Given the form of the field  $\sigma$ , (9), we infer from (45) that a nonzero value for the gap would imply, in principle, the spontaneous breakdown of the U(1) symmetry, (3). For  $N \rightarrow \infty$ , the system is effectively three-dimensional and the occurrence of a nonzero gap  $\Delta(T)$  as determined in this section is not in conflict with the theorem. For  $T = 0$ , either for finite or infinite  $N$ , also we have a non-vanishing superconducting gap, which was

studied in section 3. In both cases the Coleman-Mermin-Wagner-Hohenberg theorem does not apply and a nonzero gap leads to a nonzero order parameter according to (45).

In the case of finite  $N$  and  $T \neq 0$ , in order to comply with the theorem and yet having a superconducting phase, we may invoke the mechanism proposed by Witten [26], by means of which we may have dynamical generation of a superconducting gap without the corresponding U(1) symmetry breakdown. It goes as follows. Whenever the gap is nonzero, according to (45), we must shift the field  $\sigma$  as

$$\sigma \rightarrow \sigma - \Delta e^{i\theta} \quad (46)$$

in (8). This will produce an extra term in the effective lagrangian (8). In terms of new fermion fields, defined as

$$\hat{\psi}_{i\sigma a} \equiv e^{-i\frac{\theta}{2}} \psi_{i\sigma a}, \quad (47)$$

the extra term in (8) reads

$$\Delta \left[ \left( \hat{\psi}_{1\uparrow a}^\dagger \hat{\psi}_{2\downarrow a}^\dagger + \hat{\psi}_{2\uparrow a}^\dagger \hat{\psi}_{1\downarrow a}^\dagger \right) + \left( \hat{\psi}_{2\downarrow b} \hat{\psi}_{1\uparrow b} + \hat{\psi}_{1\downarrow b} \hat{\psi}_{2\uparrow b} \right) \right]. \quad (48)$$

This is an explicit superconducting gap term that will make

$$\langle 0 | \hat{\psi}_{1\uparrow a}^\dagger \hat{\psi}_{2\downarrow a}^\dagger + \hat{\psi}_{2\uparrow a}^\dagger \hat{\psi}_{1\downarrow a}^\dagger | 0 \rangle \neq 0. \quad (49)$$

Since the U(1) symmetry acts as  $\psi_{i\sigma a} \rightarrow e^{i\omega} \psi_{i\sigma a}$  and  $\theta \rightarrow \theta + 2\omega$  we immediately see that the field  $\hat{\psi}_{i\sigma a}$  is invariant under U(1) rotations and therefore the non-vanishing expectation value above does not imply spontaneous breakdown of the U(1) symmetry (notice that the chiral U(1) symmetry (4) is also unbroken). Thus, we can have dynamical generation of a superconducting gap without the associated spontaneous symmetry breaking [26].

We must examine the thermodynamic conditions for the occurrence of this situation. This has been done in detail for the case of the Gross-Neveu model in 2+1D [25] and also in the case of the semimetal-excitonic insulator transition that occurs in layered materials [29], both related to the potential spontaneous breakdown of the chiral symmetry. The results of these analysis also apply here.

The basic point is that, in order to check whether the order parameter (45) is zero or not, we must analyze the thermodynamics of the phase  $\theta$  of the superconducting order parameter. It turns out that this phase decouples and suffers a Kosterlitz-Thouless [27] transition at a temperature  $T_{KT}$ . For temperatures above  $T_{KT}$  there is no phase coherence and the superconducting order parameter vanishes because then  $\langle \cos \theta \rangle = \langle \sin \theta \rangle = 0$  (even though  $\Delta$  may be different from zero). Below  $T_{KT}$  there is a phase ordering and there will be a nonvanishing gap provided the condition  $T < T_c$  is also met (otherwise

$\Delta = 0$ ). As it is,  $T_{KT} \leq T_c$  [25] and, therefore, the actual superconducting transition occurs at  $T_{KT}$ . It can be shown that  $T_{KT} \xrightarrow{N \rightarrow \infty} T_c$  [25]. This clearly indicates that in spite of the fact that we may have a superconducting gap at a finite temperature in two-dimensional space, only in a really three-dimensional system we will have phase coherence developing at the same time that the modulus of the order parameter becomes nonzero, as determined by the gap equations.

It has been speculated [28] that the above scenario could provide a framework for explaining the pseudogap transition that precedes the superconducting transition in high- $T_c$  cuprates in the underdoped region. Our model provides a concrete realization of this mechanism.

## 5 Systems with a Physical Cutoff $\Lambda$

### 5.1 Physical Cutoff

When considering applications in condensed matter systems, one usually finds a natural energy cutoff  $\Lambda$  (momentum cutoff  $\Lambda/v_F$ ). The Debye frequency (energy) is an example, in the case of conventional BCS superconductors. In this case, no renormalization is needed and the coupling constant  $\lambda$  is the physical one. We investigate in this section the modifications that will occur in the superconducting electronic system under consideration when there is a physical cutoff in energy or momentum.

The two-body interaction in this case, instead of being a delta function leading to the local interaction in (1), is given, in terms of the momentum cutoff, by

$$V(\vec{k}) = \begin{cases} -\lambda & |\vec{k}| < \Lambda/v_F \\ 0 & |\vec{k}| > \Lambda/v_F \end{cases}. \quad (50)$$

In coordinate space, this corresponds to the interaction potential

$$V(\vec{r}) = -\lambda \frac{\Lambda}{2\pi v_F |\vec{r}|} J_1 \left( \frac{\Lambda |\vec{r}|}{2\pi v_F} \right), \quad (51)$$

where  $J_1$  is a Bessel function.

## 5.2 The $T = 0$ Case

We must now evaluate (15) with a finite physical momentum cutoff  $\Lambda/v_F$ . This yields

$$V_{\text{eff}}(|\sigma|) = \frac{|\sigma|^2}{\lambda} - \frac{2}{3\alpha} \left[ (|\sigma|^2 + \Lambda^2)^{\frac{1}{2}} - |\sigma|^3 - \Lambda^3 \right] \quad (52)$$

We would like to stress that we are not assuming that  $\Lambda$  is large compared to  $|\sigma|$ , rather, we are considering a completely arbitrary finite cutoff  $\Lambda$ . This is not the situation usually found in conventional BCS superconductors. However, it is likely to be found in nonconventional ones such as high-Tc cuprates. For  $\Lambda \gg |\sigma|$  (52) would reproduce (16).

The solutions of

$$V'_{\text{eff}}(|\sigma|) = 2|\sigma| \left[ \frac{1}{\lambda} - \frac{(|\sigma|^2 + \Lambda^2)^{\frac{1}{2}}}{\alpha} + \frac{|\sigma|}{\alpha} \right] = 0 \quad (53)$$

will give the gap in the present case. This admits two solutions, namely,  $\tilde{\Delta}_0 = 0$  or

$$\tilde{\Delta}_0 = \frac{\lambda\alpha}{2} \left[ \frac{\Lambda^2}{\alpha^2} - \frac{1}{\lambda^2} \right]. \quad (54)$$

The second derivative of the potential, evaluated at  $\tilde{\Delta}_0 = 0$  is

$$V''_{\text{eff}}(\tilde{\Delta}_0 = 0) = \frac{1}{\lambda} - \frac{\Lambda}{\alpha} \quad (55)$$

and we conclude that  $\tilde{\Delta}_0 = 0$  is a solution only for  $\lambda < \tilde{\lambda}_c$ , with  $\tilde{\lambda}_c = \alpha/\Lambda$ . Conversely, the second derivative of (52) evaluated at  $\tilde{\Delta}_0 \neq 0$  (given by (54)) is positive for  $\lambda > \tilde{\lambda}_c$ . As a consequence the gap now will be given by

$$\tilde{\Delta}_0 = \begin{cases} 0 & \lambda < \tilde{\lambda}_c \\ \frac{\alpha\lambda}{2} \left( \frac{1}{\tilde{\lambda}_c^2} - \frac{1}{\lambda^2} \right) & \lambda > \tilde{\lambda}_c \end{cases}, \quad (56)$$

which should be compared with (26), see Fig. 3. Again we find a quantum phase transition, now at the critical coupling  $\tilde{\lambda}_c = \alpha/\Lambda$ . Observe now that, contrary to the local case, since  $\Lambda$  is a physical parameter, the value of the quantum critical point  $\tilde{\lambda}_c$  is predicted by the theory.

Observe that, for  $\lambda > \tilde{\lambda}_c$

$$\tilde{\Delta}_0 = \alpha \left( \frac{1}{\tilde{\lambda}_c} - \frac{1}{\lambda} \right) \left( \frac{\lambda + \tilde{\lambda}_c}{2\tilde{\lambda}_c} \right) \quad (57)$$

and in the region where  $\lambda \gtrsim \tilde{\lambda}_c$ , we have  $\tilde{\Delta}_0 \simeq \Delta_0$ , with  $\Delta_0$  given by (26). This coincides with the regime where  $\Lambda \gg \tilde{\Delta}_0$  and occurs near the quantum critical point. Also, from (57) we see that the quantum phase transition is in the same universality class as the one found in Sect. 2, as it should.

### 5.3 The $T \neq 0$ Case

Let us consider now the case of systems with a physical cutoff at a finite temperature. In this case the gap equation (58) must be replaced by

$$1 = \frac{\lambda}{\alpha} \int_{\tilde{\Delta}}^{(\tilde{\Delta}^2 + \Lambda^2)^{\frac{1}{2}}} dy \tanh\left(\frac{y}{2T}\right). \quad (58)$$

Solving the integral, we find an implicit equation for the superconducting gap in the presence of a physical cutoff  $\Lambda$ , at an arbitrary temperature, namely

$$\tilde{\Delta}(T) = 2T \cosh^{-1} \left[ e^{-\frac{\alpha}{2T\Lambda}} \cosh \left[ \frac{(\tilde{\Delta}^2(T) + \Lambda^2)^{\frac{1}{2}}}{2T} \right] \right]. \quad (59)$$

Using the fact that  $\tilde{\Delta}_0$  satisfies (53), we may verify that indeed

$$\tilde{\Delta}(T) \xrightarrow{T \rightarrow 0} 2T \cosh^{-1} \left[ \frac{e^{\frac{\tilde{\Delta}_0}{2T}}}{2} \right] \xrightarrow{T \rightarrow 0} \tilde{\Delta}_0, \quad (60)$$

where  $\tilde{\Delta}_0$  is given by (56).

Let us now determine the critical temperature  $T_c$ , for the onset of superconductivity in the presence of a physical cutoff. We must have  $\tilde{\Delta}(T_c) = 0$  and therefore, from (59)

$$\cosh\left(\frac{\Lambda}{2T_c}\right) = e^{\frac{\alpha}{2T_c\Lambda}}. \quad (61)$$

This equation yields the following relation between  $T_c$  and the zero temperature gap, (56)

$$\tilde{\Delta}_0 = \left( \frac{\lambda + \tilde{\lambda}_c}{2\tilde{\lambda}_c} \right) 2T_c \ln \left[ \frac{2}{1 + e^{-\frac{\Lambda}{T_c}}} \right], \quad (62)$$

where  $\tilde{\lambda}_c = \alpha/\Lambda$ .

A particular regime that is frequently studied is the one where the cutoff is large compared to the critical temperature and to  $\tilde{\Delta}_0$ , namely, when  $\Lambda \gg$



$T_c$  and  $\lambda \gtrsim \tilde{\lambda}_c$ . Observe that this last condition guarantees that  $\Lambda \gg \tilde{\Delta}_0$ , according to (56) and (57). Since the previous relations imply

$$T_c \ll \Lambda = \frac{\alpha}{\tilde{\lambda}_c} \simeq \frac{\alpha}{\lambda}, \quad (63)$$

we may infer that the conditions above hold in the weak coupling regime. In this case, (62) becomes simply  $\tilde{\Delta}_0 = 2 \ln 2 \ T_c$  [12]. Also, according to (56) and (57)  $\tilde{\Delta}_0 \rightarrow \Delta_0$  in this regime. Thus we recover (42), the relation previously found between  $T_c$  and the zero temperature gap in systems without a natural cutoff. This relation gives the ratio  $\tilde{\Delta}_0/T_c \sim 1.39$ , which should be compared with the corresponding ratio in the BCS theory for conventional superconductors, namely, 1.76, which is also derived in the limit  $\Lambda/T_c \gg 1$ .

In the weak coupling regime, given by (63), we also recover expressions (43) and (44), for the superconducting gap as a function of the temperature. The pre-factor in the latter expression, describing the behavior of the gap around  $T_c$  is 2.36, whereas the corresponding value in BCS theory is 3.06.

We solve (59) numerically for  $\tilde{\Delta}$ , using different values for the ratio  $\lambda/\tilde{\lambda}_c$  and display the result in Fig. 4. In this, we may observe that indeed in the weak coupling regime the ratio  $\Delta_0/T_c$  approaches the result given by (42). On the other hand, as the coupling increases, we see that this ratio surpasses the value obtained in BCS theory and approaches the values obtained in strongly coupled systems.

There are condensed matter systems for which the weak coupling condition (63) is not valid. Should one of such systems contain Dirac fermions, we should use (56) and (59) for the superconducting gap, respectively at  $T = 0$  and  $T \neq 0$ . The critical temperature, by its turn, would be given by (62).

It is remarkable that the expression we find for the superconducting gap in the case of Dirac fermions, strongly differs from the one obtained in BCS theory in any dimensions. There the gap has an exponential dependence on the inverse of the coupling  $\lambda$ . Here, in spite of still being non-analytical in the coupling constant  $\lambda$ , the gap has a power-law dependence on it and vanishes below a critical value at zero temperature. The different behavior can be traced back to the fact that in the case of Dirac fermions the density of states vanishes at the Fermi points. In the BCS case, however, we have a Fermi surface with a non-vanishing density of states at the Fermi level and the momentum integrals used for obtaining the effective potential may be evaluated as

$$\int \frac{d^2 k}{(2\pi)^2} \simeq N(\epsilon_F) \int_{-\Lambda}^{\Lambda} d\xi, \quad (64)$$

where  $N(\epsilon_F)$  is the density of states at the Fermi level. This leads to a gap proportional to  $e^{-\frac{1}{N(\epsilon_F)\lambda}}$ .

## 6 Conclusions

Our results highlight the qualitative and quantitative differences existing between Dirac electrons – for which the Fermi surface reduces to a point – and quasi-free electrons, having a dispersion relation  $\epsilon(\vec{k}) = |\vec{k}|^2/2m^*$ , which leads to a Fermi surface formation. The properties of the former are analyzed when the conditions are such that a superconducting interaction is present in a quasi-two-dimensional system consisting of a stack of  $N$  planes. One of the most striking differences between the two electron types, is the polynomial, rather than exponential gap dependence on the inverse coupling constant, in the case of Dirac electrons. This leads to a quantum phase transition separating a normal from a superconducting phase for a critical value of the coupling, at  $T = 0$ . This transition is rather similar to the one occurring in the Nonlinear Sigma model in 2+1D, where the spin stiffness has an expression identical to (26) [30]. Because of this quantum phase transition, our model reproduces qualitatively the superconducting transition in high- $T_c$  cuprates in the underdoped regime as one can infer from Fig. 2. This seems to indicate that Dirac electrons may have an important role in the mechanism of high- $T_c$  superconductivity.

The ratio between the zero temperature gap to  $T_c$ , as well as the behavior of the gap around  $T_c$ , are results also significantly different for the case of Dirac electrons. It would be very interesting to compare our results with experimental measurements of these quantities in quasi-two-dimensional materials containing Dirac electrons.

An interesting outcome of this work is the analysis the phase structure generated by the model in the presence of a natural physical cutoff, specially the regime where the cutoff is of the same order of the gap and we are away from the quantum critical point. In this case the (naturally) cutoff theory yields results that are strongly different from the ones derived in the weak coupling regime where the cutoff is much larger than the gap and the system is close to the quantum critical point. The strong coupling regime, in particular, is probably relevant for the high- $T_c$  cuprates.

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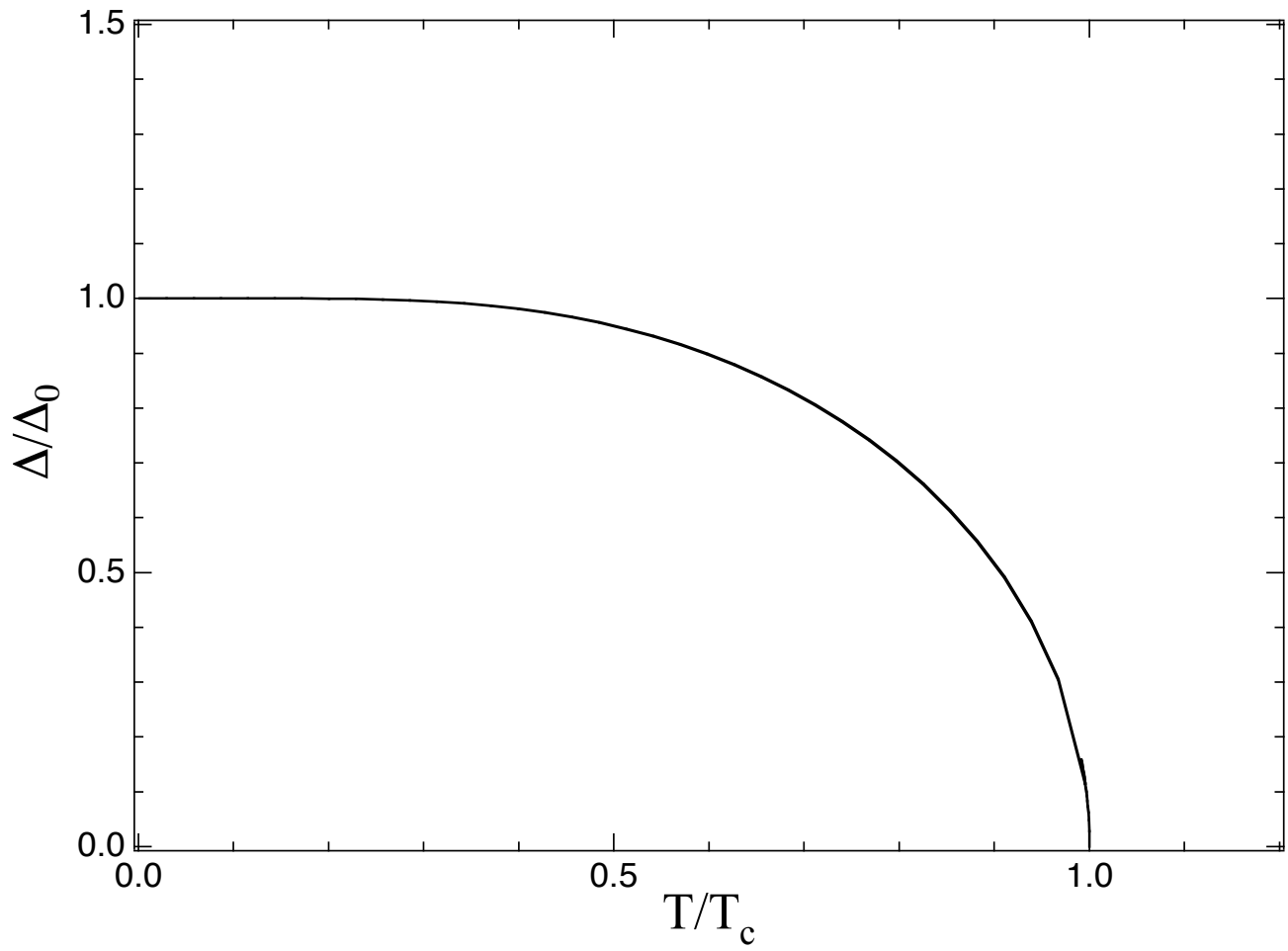


Fig. 1. The normalized superconducting gap  $\Delta/\Delta_0$  as a function of the normalized temperature  $T/T_c$ .

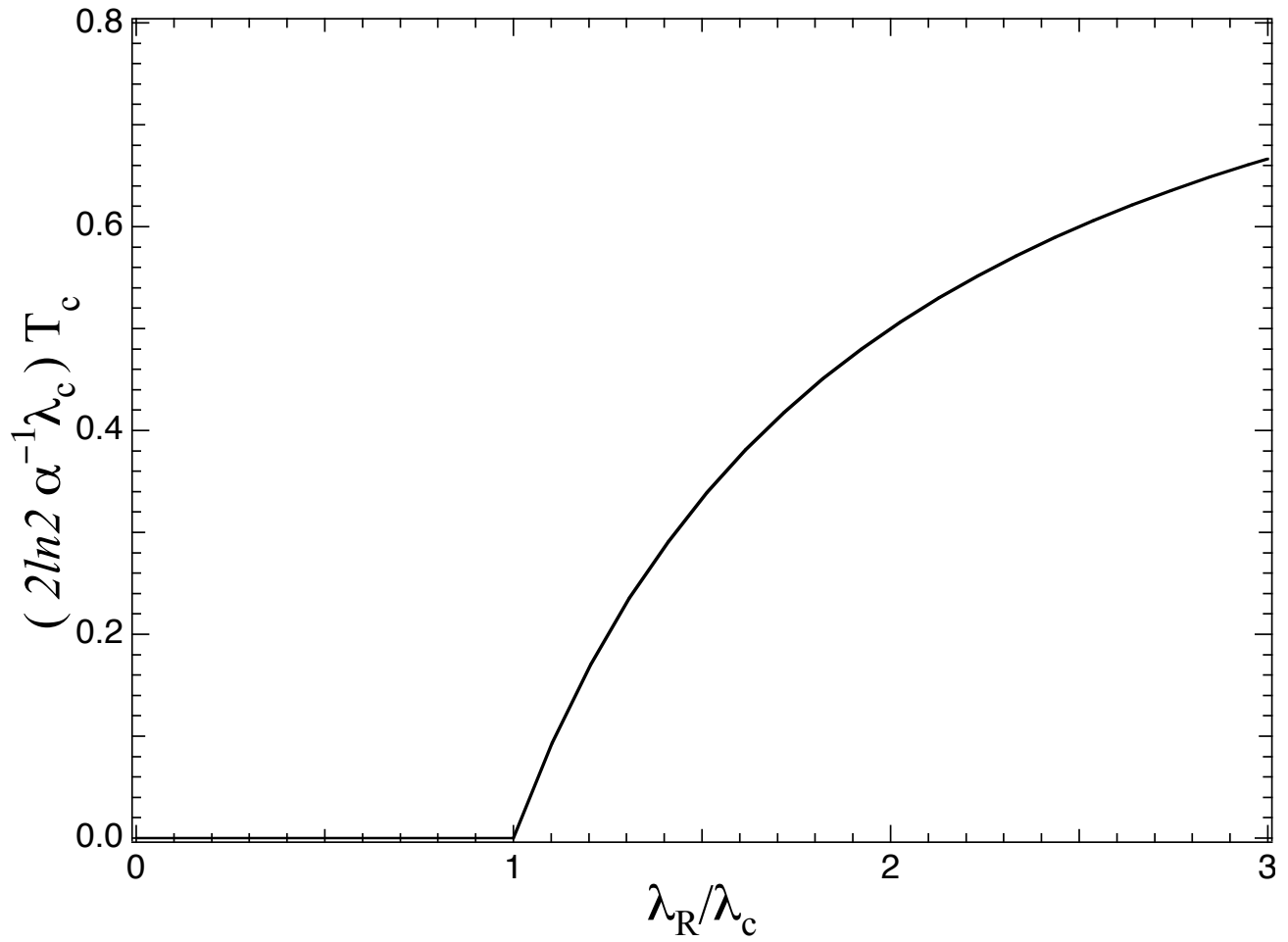


Fig. 2. The superconducting critical temperature  $T_c$  as a function of the renormalized coupling  $\lambda_R$ .

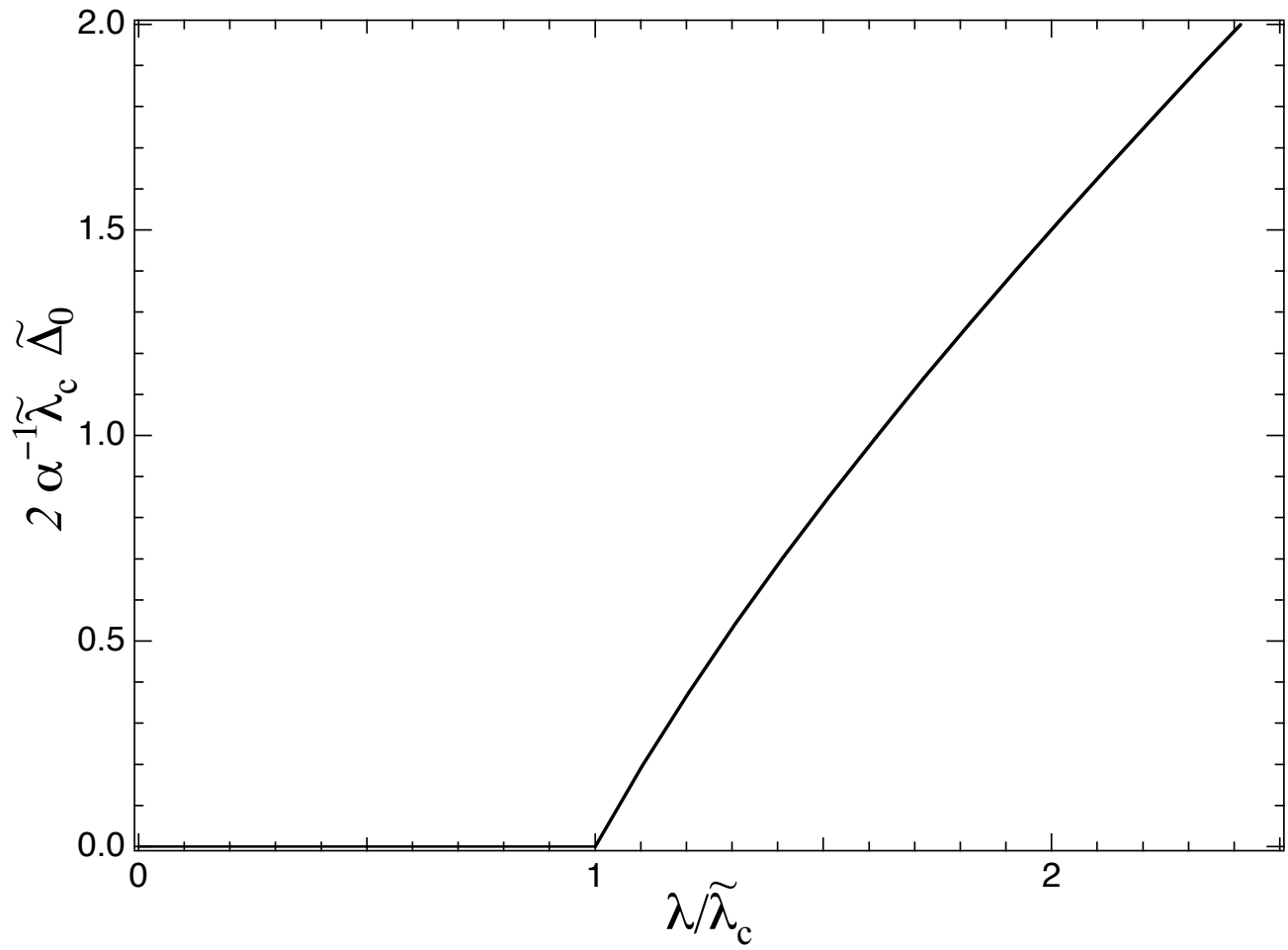


Fig. 3. The zero temperature superconducting gap as a function of the normalized coupling  $\lambda/\tilde{\lambda}_c$ .

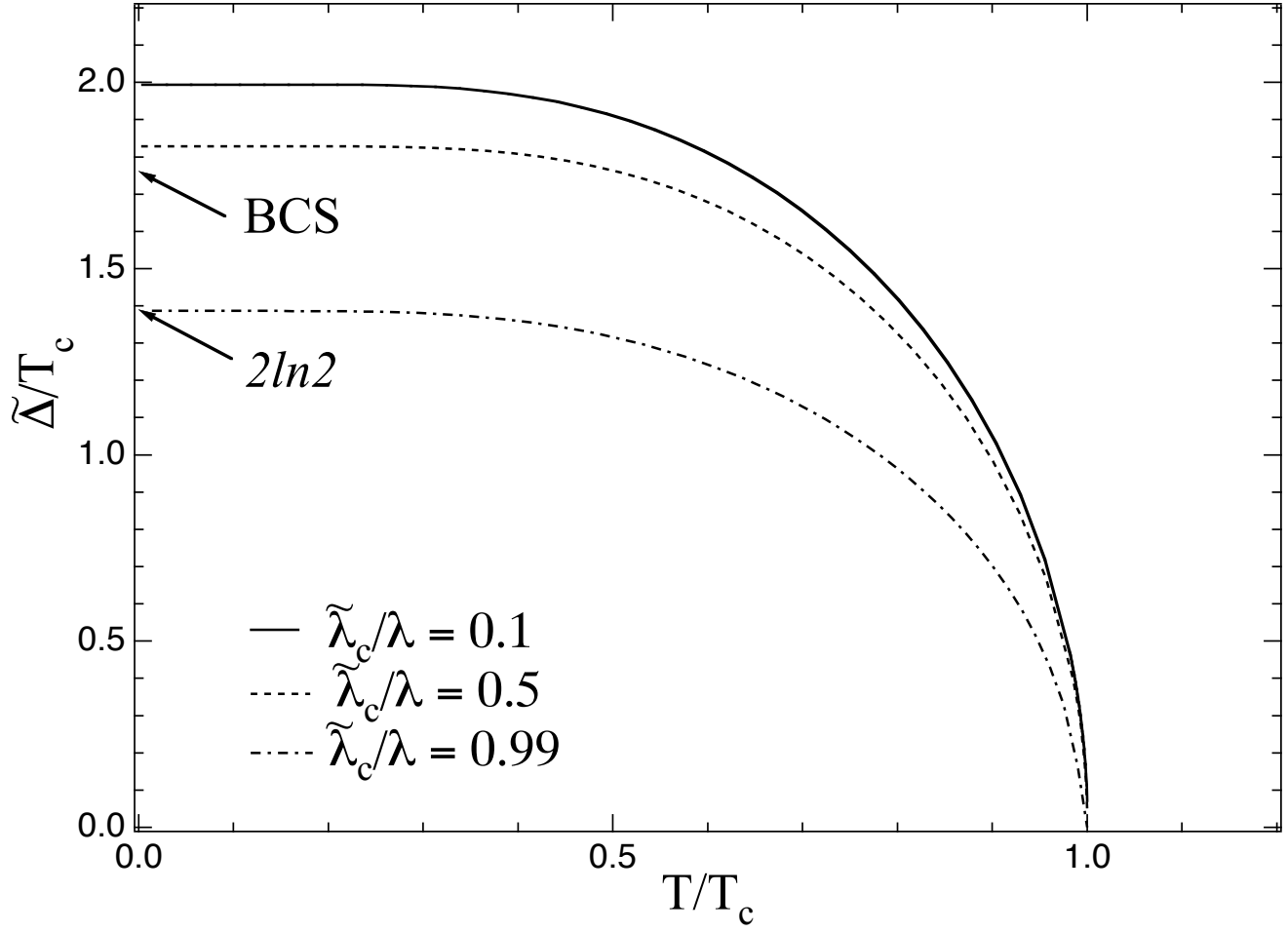


Fig. 4. The superconducting gap  $\tilde{\Delta}(T)$  divided by  $T_c$  as a function of the normalized temperature  $T/T_c$  for several values of  $\tilde{\lambda}_c/\lambda$ . The two arrows indicate the values for  $\tilde{\Delta}_0/T_c$  given by the BCS theory and the  $\lambda \simeq \tilde{\lambda}_c$  case ( $\tilde{\lambda}_c = \alpha/\Lambda$ ).